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ABSTRACT

Solitary waves in running polytropic gases of finite or infinite height are studied. Explicit expressions for the critical speed and the solution of the solitary waves are obtained by a perturbation scheme applied to the non-linear equations. It is found that internal waves may also be observed in a running gas, and change of wave type may occur in the medium. The results obtained are expressed in terms of simple integrals of density and velocity distributions of the medium in the state of equilibrium, and conform with solutions for liquids of constant density as special cases.

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1. Introduction.

Since 1834 when Scott Russel made his first observation of a solitary wave on a stream, much theoretical work has been done on the problem of solitary waves in a liquid of constant density. We refer the reader to Stoker's book, "Water Waves" [1], for a detailed discussion of the problem and the references cited there. In recent years, there has been growing interest in the study of solitary waves in a stratified medium. An interesting account of the recent developments of the solitary wave problem has been given in [2] (Peters and Stoker, 1960). A variant of Friedrichs' procedure is used by Peters and Stoker to solve the problem of solitary waves in liquids having non-constant density, and the so-called "internal waves" are found on the interface of a two-layer liquid. Recently, Benjamin also solved the problem of solitary waves on a stream with an arbitrary distribution of vorticity [3]. In all the previous work, the medium considered is always assumed to be incompressible, and the height of the medium finite. For the work done here, we shall be concerned with the problem of solitary waves in a running polytropic gas of finite or infinite height. The main difficulty encountered in our problem is that the density as well as the pressure vanishes on the free surface, which may be at infinity, and the equations become singular there. In order to overcome this difficulty, we use a transformation established by Peters and Stoker to express all the functions in terms of the horizontal distance x and

the stream function ψ as new independent variables, and the limiting values of all the functions as the image of the free boundary in the x,ψ-plane is approached from below are then defined as the values of the function there. Furthermore, since ψ is a function of vertical distance ζ in the state of equilibrium, we may eventually formulate the problem in the x,ζ-plane as Benjamin first did for the incompressible case. However, the approach applied here is different from that in [3].

We formulate our problem in §2. The solitary wave solution is obtained in §3. The coefficients of the solitary wave equation are reduced to simple integrals of the density distribution ρ_∞ and the velocity distribution u_∞ in the state of equilibrium. It is shown in §4 that the known results for liquids of constant density are special cases of our solution. Furthermore, the so-called "neutral stream lines" are found and some criteria for the existence of such lines are established. As a consequence, the internal waves may also be observed in a running gas. The appendices contain the details of derivations which we have omitted in the main text. The results obtained here are only for the case of a polytropic gas with arbitrary equilibrium velocity profile, but it is feasible to extend the method to gases with more general pressure and density relations. We hope that the research reported here, if relevant, may be of some use to the study of certain geophysical phenomena in the upper atmosphere.

This report is intended to supersede Part I and Part III of an earlier report [4], the results of which are only special cases of the solution given below. At the same time, we also take this opportunity to correct a few errors committed in the computation of the coefficients of the solitary wave equation given in [4].

Solitary Wave in a Running Gas

2. Formulation of the Problem.

We consider a layer of a polytropic compressible medium supported by a rigid plane bottom and having a free surface which may or may not be at infinity. The pressure at the free surface is assumed to be zero and there are no geometric constraints. In the state of equilibrium the velocity profile as a function of the vertical distance only is assumed to be given. A cross section of the medium is a horizontal strip in the upper half plane (Fig. 1). It is supposed that a wave of permanent type moving to the left with constant velocity c has been generated in the medium as a result of some disturbance. We choose a coordinate system moving with the wave such that the x-axis coincides with the bottom and the y-axis passes through the crest of a wave of elevation or the trough of a wave of depression and is positive upward (Fig. 1). With respect to the coordinate system the wave is stationary and the medium at infinity is in the state of equilibrium.

The equations governing a two-dimensional steady flow

of a polytropic compressible medium are

$$
(2.1)\qquad
\begin{aligned}
&\frac{\partial(\tilde{\rho}\tilde{u})}{\partial x} + \frac{\partial(\tilde{\rho}\tilde{v})}{\partial y} = 0 \\
&\tilde{u}\,\frac{\partial\tilde{u}}{\partial x} + \tilde{v}\,\frac{\partial\tilde{u}}{\partial y} = -\,\frac{1}{\tilde{\rho}}\,\frac{\partial\tilde{p}}{\partial x} \\
&\tilde{u}\,\frac{\partial\tilde{v}}{\partial x} + \tilde{v}\,\frac{\partial\tilde{v}}{\partial y} = -\,g - \frac{1}{\tilde{\rho}}\,\frac{\partial\tilde{p}}{\partial y} \\
&\frac{\tilde{p}}{\tilde{p}_0} = \left(\frac{\tilde{\rho}}{\tilde{\rho}_0}\right)^n , \qquad n \geq 1,
\end{aligned}
$$

where $\tilde{p}(x,y)$ and $\tilde{\rho}(x,y)$ are respectively the pressure and density, $\tilde{u}(x,y)$, $\tilde{v}(x,y)$ the horizontal and the vertical velocity component, g is the gravitational constant, and $\tilde{p}_0$, $\tilde{\rho}_0$ are the reference pressure and density. In the state of equilibrium, we have

$$
(2.2)\qquad
\begin{aligned}
\tilde{\rho} &= \tilde{\rho}_0\left[1 - \frac{n-1}{n}\,\frac{\zeta}{h}\right]^{1/n-1}, \\
&\qquad \text{for } n > 1,\ 0 \leq \zeta \leq \zeta_s = \frac{n}{n-1}\,h, \\
&= \tilde{\rho}_0\, e^{-\zeta/h} \\
&\qquad \text{for } n = 1,\ 0 \leq \zeta \leq \zeta_s = \infty ,
\end{aligned}
$$

where $h = \tilde{p}_0/g\tilde{\rho}_0$, ζ is the vertical distance at equilibrium and , $\tilde{\rho}_0$ is the value of $\tilde{\rho}_\infty$ at $\zeta = 0$.

Denote by $\tilde{\psi}(x,y)$ the stream function such that

$$
\tilde{\psi}_y = \tilde{\rho}\tilde{u} , \qquad \tilde{\psi}_x = -\,\tilde{\rho}\tilde{v} .
$$

The mass flux across any vertical plane from $y = 0$ to the free surface $y = y_s$ per unit breadth is given by

$$\tilde{\psi}(x,y_s) - \tilde{\psi}(x,0) = \int_0^{y_s} \tilde{\rho}_\infty \tilde{u}_\infty \, dy = \gamma_s ,$$

where we may set $\tilde{\psi}(x,0) = 0$.

Let

$$\tilde{\psi}(x,y) = \gamma$$

where $0 \leq y < y_s$, $0 \leq \gamma < \gamma_s$, and assume that for any value of γ in $0 \leq \gamma < \gamma_s$ there exists a unique solution of the above equation for y such that

$$y = \bar{f}(x,\gamma) .$$

Here the bar notation is used to denote a quantity as a function of x and γ. If we choose $\bar{f}$, $\bar{p}$, $\bar{\rho}$, and $\bar{u}$ as dependent variables, then we have from (2.1), for $0 \leq \gamma < \gamma_0$, $-\infty < x < \infty$,

$$\begin{aligned} &\bar{u}_x = \bar{f}_\gamma \bar{p}_x + \bar{f}_x \bar{p}_\gamma \\ &\bar{u}\bar{f}_{xx} + \bar{u}_x \bar{f}_x = - \bar{\rho} g \bar{f}_\gamma - \bar{p}_\gamma \\ &\bar{\rho}\bar{u}\bar{f}_\gamma = 1, \qquad \bar{v} = \bar{u}\bar{f}_x \\ &\frac{\bar{p}}{\tilde{p}_0} = \left(\frac{\bar{\rho}}{\tilde{\rho}_0}\right)^n , \qquad n \geq 1, \end{aligned} \tag{2.3}$$

subject to the boundary conditions

$$\bar{f}(x,0) = 0, \qquad \bar{p}(x,\gamma_s) = 0.^{(1)}$$

---

(1) It is noted here that the transformation from x,y-plane to x,γ-plane breaks down as $\gamma=\gamma_s$ since $\bar{f}_\gamma=\infty$ there. Hence we define the limit of a functionas $\gamma \to \gamma_s$ as the value of the function at $\gamma = \gamma_s$. In fact, $\gamma = \gamma_s$ is a branch line in the x,γ-plane. For a general discussion of the mappings, c.f. [5].

However, in order to make $\tilde{u}_\infty$ and $\tilde{\rho}_\infty$ appear explicitly in the above equations, it will be more convenient to use the vertical distance ζ at equilibrium as an independent variable. The relation between γ and ζ is given by

$$\gamma = \int_0^\zeta \tilde{\rho}_\infty \tilde{u}_\infty \, dy ,$$

and assuming that $d\gamma/d\zeta > 0$, for $0 \leq \zeta < \zeta_s$, we have

$$\frac{\partial \bar{\phi}}{\partial \gamma} = \frac{1}{\hat{G}_\infty} \frac{\partial \phi}{\partial \zeta} ,$$

where

$$\frac{d\gamma}{d\zeta} = \hat{G} = \tilde{\rho}_\infty(\zeta)\tilde{u}_\infty(\zeta) ,$$

and the hat notation denotes a function of x,ζ. Since $\tilde{\rho}_\infty(\zeta)$ is always positive for $0 \leq \zeta < h_s$ where $h_s = \bar{f}(x,\gamma_s)$, we suppose that

$$\tilde{u}_\infty(\zeta) \neq 0$$

for $0 \leq \zeta \leq \zeta_s$.

By using x,ζ as independent variables (2.3) becomes, for $0 < \zeta < \zeta_s$, $-\infty < x < \infty$,

$$(2.4)\qquad \begin{aligned} &\hat{G}_\infty \hat{u}_x = - \hat{f}_\zeta \hat{p}_x + \hat{f}_x \hat{p}_\zeta \\ &\hat{G}_\infty(\hat{u}\hat{f}_{xx} + \hat{u}_x \hat{f}_x) = - \hat{\rho} g \hat{f}_\zeta - \hat{p}_\zeta \\ &\hat{\rho}\hat{u}\hat{f}_\zeta = \hat{G}_\infty , \qquad\qquad \hat{v} = \hat{u}\hat{f}_x, \\ &\frac{\hat{p}}{\tilde{p}_0} = \left(\frac{\hat{\rho}}{\tilde{\rho}_0}\right)^n , \qquad\qquad n \geq 1 , \end{aligned}$$

subject to the boundary conditions

$$\hat{f}(x,0) = 0 , \qquad \hat{p}(x,\zeta_s) = 0 .$$

Now we introduce the following dimensionless variables

$$\xi = \frac{x}{h} , \qquad v = \frac{\hat{v}}{c} , \qquad u = \frac{\hat{u}}{c} ,$$

$$\eta = \frac{\zeta}{h} , \qquad p = \frac{\hat{p}}{\tilde{\rho}_0 c^2} , \qquad \rho = \frac{\hat{\rho}}{\tilde{\rho}_0} ,$$

$$f = \frac{\hat{f}}{h} , \qquad \lambda = \frac{gh}{c^2} , \qquad G_\infty = \frac{\hat{G}_\infty}{\tilde{\rho}_\infty c} ,$$

$$\rho_\infty = \frac{\tilde{\rho}_\infty}{\tilde{\rho}} , \qquad u_\infty = \frac{\tilde{u}_\infty}{c} , \qquad \eta_s = \frac{\zeta_s}{h} ,$$

where $h = \tilde{p}_0/g\tilde{\rho}_0$, and (2.4) become, for $0 < \eta < \eta_s$, $-\infty < \xi < \infty$,

$$\begin{aligned} &G_\infty u_\xi = - f_\eta p_\xi + f_\xi p_\eta \\ &G_\infty(u f_{\xi\xi} + u_\xi f_\xi) = - \rho\lambda f_\eta - p_\eta \\ &\rho u f_\eta = G , \qquad\qquad v = u f_\xi , \\ &p = \lambda \rho^n , \qquad\qquad n \geq 1 , \end{aligned}$$

subject to the boundary conditions

$$f(\xi,0) = 0 , \qquad p(\xi,\eta_s) = 0 .$$

3. <u>Nonlinear Theory. Solitary Wave Solution</u>.

It is assumed that a solitary wave moves with a speed such that $\lambda = gh/c^2$ is near some positive value ℓ, which will be determined later. The equations (2.6) can be written as, for $0 < \eta < \eta_s$, $-\infty < \xi < \infty$,

$$
\begin{aligned}
&G_\infty u_\xi = - f_\eta p_\xi + f_\xi p_\eta \\
&G_\infty(uf_{\xi\xi} + u_\xi f_\xi) = (\ell-\lambda)\rho f_\eta - \ell\rho f_\eta - p_\eta \\
(3.1)\qquad &\rho u f_\eta = G_\infty , \qquad v = uf_\xi , \\
&p = -(\ell-\lambda)\rho^n + \ell\rho^n , \quad n \geq 1 .
\end{aligned}
$$

We let

$$\varepsilon = \ell - \lambda$$

and introduce a new variable

$$\sigma = \xi\sqrt{\varepsilon}$$

Then in terms of η, σ, (3.1) become, for $0 < \eta < \eta_s$, $-\infty < \sigma < \infty$,

$$
\begin{aligned}
&G_\infty u_\sigma = - f_\eta p_\sigma + f_\sigma p_\eta \\
(3.2)\qquad &\varepsilon G_\infty(uf_{\sigma\sigma} + u_\sigma f_\sigma) = \varepsilon\rho f_\eta - \ell\rho f_\eta - \;{}_{\eta} \\
&\rho u f_\eta = G_\infty , \qquad v = \varepsilon u f_\sigma , \\
&p = - \varepsilon\rho^n + \ell\rho^n , \qquad n \geq 1,
\end{aligned}
$$

subject to the boundary conditions

$$f(\sigma,0) = 0 , \qquad p(\sigma,\eta_s) = 0.$$

Suppose that all the dependent variables can be expanded

in integral powers of ε, i.e.

(3.3) $$\phi(\sigma,\eta,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \phi_k(\sigma,\eta) \ ,$$

where ϕ stands for p, ρ, u and f. Substitution of (3.3) in (3.2) will yield a sequence of equations and boundary conditions for the successive approximations by letting the coefficients of like powers of ε be equal.

The equations for the zero-th order approximation are, for $0 < \eta < \eta_s$, $-\infty < \sigma < \infty$,

(3.4) $$\begin{aligned} G_\infty u_{0\sigma} &= - p_{0\sigma} f_{0\eta} + f_{0\sigma} p_{0\eta} \\ 0 &= - \ell \rho_0 f_{0\eta} - p_{0\eta} \\ \rho_0 u_0 f_{0\eta} &= G_\infty \\ p_0 &= \ell \rho_0^n \ , \quad n \geq 1, \end{aligned}$$

subject to

$$f_0(\sigma,0) = 0 \ , \qquad p(\sigma,\eta_s) = 0.$$

(3.4) is a set of non-linear equations, the solution of which cannot be found by any standard method. However, we may assume $\rho_0 = \rho_\infty$ since the solitary wave solution is essentially a small perturbation over the equilibrium state at $\sigma = -\infty$. Based upon this ansatz, we find the solution for the zero-th approximation as follows.

$$
(3.5) \qquad
\begin{aligned}
p_0 &= \ell \rho_\infty^n , \qquad\qquad n \geq 1, \\
\rho_0 &= \rho_\infty \quad , \\
f_0 &= \eta \quad , \\
u_0 &= u_\infty \ .
\end{aligned}
$$

The equations for the first order approximation are, $0 < \eta < \eta_s$, $-\infty < \sigma < \infty$,

$$
(3.6) \qquad
\begin{aligned}
G_\infty u_{1\sigma} &= - f_{0\eta} p_{1\sigma} + f_{1\sigma} p_{0\eta} , \\
0 &= \rho_0 f_{0\rho} - \ell(\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - p_{1\eta} , \\
\rho_1 u_0 f_{0\eta} &+ \rho_0 u_1 f_{0\eta} + \rho_0 u_0 f_{1\eta} = 0 , \\
p_1 &= \ell n \rho_1 \rho_0^{n-1} - \rho_0^r
\end{aligned}
$$

subject to

$$
f_1(\sigma,0) = 0 , \qquad\qquad p_1(\sigma,\eta_s) = 0 ,
$$

where use is made of the fact that p_0, ρ_0, f_0 and u_0 are functions of η only. Elimination of f_1, ρ_1, and u_1 from (3.6) and making use of (3.5) yield the equation

$$
(3.7) \qquad (u_\infty^2 p_{1\eta\sigma})_\eta = \frac{\rho_\infty'}{\rho_\infty} (u_\infty^2 p_{1\eta\sigma}) .
$$

The derivation of (3.7) is deferred to Appendix I. From (3.7) we have

$$
(3.8) \qquad p_{1\eta\sigma} = - \frac{\rho_\infty}{u_\infty^2} \ell a'(\sigma)
$$

and since $p_{1\sigma}(\sigma,\eta_s) = 0$, we also obtain

$$p_{1\sigma} = \ell a'(\sigma)F(\eta) \ , \tag{3.9}$$

where $a'(\sigma)$ is an arbitrary function of σ and

$$F(\eta) = \int_{\eta}^{\eta_s} \frac{\rho_\infty(\eta')}{u_\infty^2(\eta')}\, d\eta' \ .$$

From the last equation of (3.6) we have

$$\begin{aligned}\rho_{1\sigma} &= \frac{1}{\ell n}\,\rho_0^{-n+1}p_{1\sigma}\\ &= \frac{1}{n}\,\rho_0^{-n+1}a'(\sigma)F(\eta) \ ,\end{aligned}$$

and

$$\rho_1 = \frac{1}{n}\,\rho_0^{-n+1}a(\sigma)F(\eta) + b(\eta)$$

where we assume $a(\sigma) \not\equiv 0$. Since ρ tends to ρ_∞ as $\sigma \to -\infty$ and we also choose $a(-\infty) = 0$, the above equation shows that $b(\eta) = 0$ and

$$\rho_1 = \frac{1}{n}\,\rho_0^{-n+1}a(\sigma)F(\eta).$$

It follows that

$$\begin{aligned}p_1 &= \ell n\rho_1\rho_0^{n-1} - \rho_0^n\\ &= \ell a(\sigma)F(\eta) - \ell^{-1}p_0 \ .\end{aligned}$$

From (3.5), (3.6), (3.8) and (3.9), we find that

$$\begin{aligned}f_{1\sigma} &= p_0\eta^{-1}[\ell^{-1}u^2p_{1\eta\sigma} + f_{0\eta}p_{1\sigma}]\\ &= -\,a'(\sigma)[-\ell^{-1} + \rho_\infty^{-1}F(\eta)] \ ,\end{aligned}$$

and

$$f_1 = -\,a(\sigma)[-\ell^{-1} + \rho_\infty^{-1}F(\eta)] + b_1(\eta) \ .$$

Since $f_1 \to 0$ as $\sigma \to -\infty$, we have $b_1(\eta) = 0$ and

$$f_1 = - a(\sigma)[-\ell^{-1} + \rho_\infty^{-1} F(\eta)] \ .$$

By applying the condition $f_1(\sigma, 0) = 0$, we see that

$$\ell = F^{-1}(0) \ . \tag{3.10}$$

In summary, the solutions for the first order approximation are

$$\begin{aligned} p_1 &= \ell a(\sigma) F(\eta) - \ell^{-1} p_0 \ , \\ \rho_1 &= \frac{1}{n} \rho_0^{-n+1} a(\sigma) F(\eta) \ , \\ f_1 &= - a(\sigma)[-\ell^{-1} + \rho_\infty^{-1} F(\eta)] \ , \\ u_1 &= - \frac{a(\sigma)}{u_\infty} \ . \end{aligned} \tag{3.11}$$

To determine the function $a(\sigma)$ we must proceed to the equations for the second order approximation, which are, for $0 < \eta < \eta_s$, $-\infty < \sigma < \sigma$,

$$\begin{aligned} G_\infty u_{2\sigma} &= - (p_{2\sigma} f_{0\eta} + p_{1\sigma} f_{1\eta}) + (f_{2\sigma} p_{0\eta} + f_{1\sigma} p_{1\eta}) \ , \\ G_\infty (u_\infty f_{1\sigma\sigma}) &= (\rho_0 f_{1\eta} + {}_1 f_{0\eta}) - \ell(\rho_0 f_{2\eta} + \rho_1 f_{1\eta} + \rho_2 f_{0\eta}) - p_{2\eta} \ , \\ \rho_0 u_2 d_{0\eta} + \rho_0 u_0 f_{2\eta} + \rho_2 u_0 f_{0\eta} &+ \rho_0 u_1 f_{1\eta} + \rho_1 u_0 f_{1\eta} + \rho_1 u_1 f_{0\eta} = 0 \ , \\ p_2 &= \ell n \rho_0^{n-1} \rho_2 - n \rho_1 \rho_0^{n-1} + \frac{n(n-1)}{2} \ell \rho_1^2 \rho_0^{n-2} \ , \end{aligned} \tag{3.12}$$

subject to the conditions

$$f_2(\sigma, 0) = 0 \ , \qquad p_2(\sigma, \eta_s) = 0 \ .$$

From the solutions obtained for the zero-th and first approximations, we can obtain the following equations

$$(3.13)\qquad \begin{aligned} \left(\frac{u_\infty^2 p_{2\eta\sigma}}{\rho_\infty}\right)_\eta &= \frac{1}{\rho_\infty}\, g_4(\sigma,\eta)\ , \\ f_{2\sigma} &= (p_{0\eta})^{-1}(\ell^{-1}u_\infty^2 p_{2\eta\sigma} + p_{2\sigma}f_{0\eta} + \ell^{-1}g_{2\sigma} - g_1)\ , \end{aligned}$$

where

$$g_1(\sigma,\eta) = -\, p_{1\sigma}g_{1\eta} + f_{1\sigma}p_{1\eta}\ ,$$

$$g_2(\sigma,\eta) = G_\infty u_\infty^3 f_{1\sigma\sigma} + G_\infty u_1 + \ell\rho_\infty u_1^2\ ,$$

$$g_3(\sigma,\eta) = -\, u_\infty^2 \rho_1 f_{1\eta} + \rho_\infty u_1^2\ ,$$

$$\begin{aligned} g_4(\sigma,\eta) = &-\, g_{2\eta\sigma} - \rho_0^{-1}\rho_{1\sigma}p_{0\eta} + \ell(n-1)p_{0\eta}\rho_1\rho_{1\sigma}\rho_0^{-2} \\ &+ p_{0\eta}\rho_0^{-1}(-u_\infty^{-2}g_{2\sigma} + \ell u_\infty^{-2}g_{3\sigma}) \\ &+ p_{0\eta\eta}p_{0\eta}^{-1}(g_{2\sigma} - \ell g_1) + \ell g_{1\eta}\ . \end{aligned}$$

The derivation of (3.13) will be given in Appendix I.

Now using the condition $f_{2\sigma}(\sigma,0) = 0$, we obtain from (3.10) and (3.13) that

$$(3.14)\quad F(0)u_\infty^2(0)p_{2\eta\sigma}(\sigma,0) + p_{2\sigma}(\sigma,0) + F(0)g_{2\sigma}(\sigma,0) - g_1(\sigma,0) = 0.$$

In order to eliminate $p_{2\eta\sigma}(\sigma,0)$ and $p_{2\sigma}(\sigma,0)$ from (3.14), we must make use of the first equation of (3.13). Multiplying both sides of the equation by $F(\eta)$ and then integrating with respect to η from 0 to η_s, we have, by integration by parts,

$$\left[F(\eta)\,\frac{u_\infty^2 p_{2\sigma\eta}}{\rho_\infty} + p_{2\sigma}\right]_0^{\eta_s} = \int_0^{\eta_s} F(\eta')\,\frac{g_4(\sigma,\eta')}{\rho_\infty(\eta')}\,d\eta'\ .$$

It can be shown that $F(\eta)u_\infty^2\rho_\infty^{-1}p_{2\sigma\eta} \to 0$ as $\eta \to \eta_s$ if $u_\infty(\eta_s) \neq 0$, and since $p_{2\sigma}(\sigma,\eta_s) = 0$, the above equation becomes

$$- F(0)u_\infty^2(0)p_{2\sigma\eta}(\sigma,0) - p_{2\sigma}(\sigma,0) = \int_0^{\eta_s} F(\eta')g_4(\sigma,\eta')\rho_\infty^{-1}(\eta')\, d\eta' .$$

It follows from (3.14) that

$$-\int_0^{\eta_s} F(\eta')g_4(\sigma,\eta')\rho_\infty^{-1}(\eta')\, d\eta' + F(0)g_{2\sigma}(\sigma,0) - g_1(\sigma,0) = 0 .$$

This gives the condition to determine the function $a(\sigma)$. By some lengthy but straightforward calculations, we finally obtain the equation

$$m_0 a'''(\sigma) + m_1 a'(\sigma)a(\sigma) + m_2 a'(\sigma) = 0$$

where

$$m_0 = -\int_0^{\eta_s} \ell u_\infty^2(\eta)\rho_\infty(\eta)(\ell^{-1} - \rho_\infty^{-1}(\eta)F(\eta))^2\, d\eta ,$$

(3.16)
$$m_1 = -3\int_0^{\eta_s} \ell\rho_\infty(\eta)u_\infty^{-4}(\eta)\, d\eta - \rho_\infty'(0)F(0) ,$$

$$m_2 = F(0) ,$$

$$\ell = F^{-1}(0) .$$

The derivation of these results will be given in Appendix II. Suppose that none of m_0, m_1 and m_2 are equal to zero, and we also impose the conditions

$$a'(-\infty) = a''(-\infty) = a'(0) = 0 ,$$

Then the solution for $a(\sigma)$ is

$$a(\sigma) = - \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{\sigma}{2} \sqrt{- \frac{m_2}{m_0}} \quad .$$

Assume that the successive approximations up to the first order give a sufficiently accurate representation of a solitary wave. We have, in terms of the independent variables x, and ζ, for $0 < \zeta < \eta_s$, $-\infty < x < +\infty$,

$$\hat{p} \cong \tilde{p}_0 \rho_\infty^n - (\ell-\lambda)\tilde{p}_0 F(\eta) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \quad ,$$

$$\hat{\rho} \cong \tilde{\rho}_0 \rho_\infty - (\ell-\lambda) n^{-1} \tilde{\rho}_0 \rho_\infty^{-n+1} F(\eta) \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \quad ,$$

$$\hat{f} \cong h\eta - (\ell-\lambda) h[\ell^{-1} - \rho_\infty^{-1} F(\eta)] \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \quad ,$$

$$(3.17) \quad \hat{u} \cong cu_\infty + (\ell-\lambda) c(u_\infty)^{-1} \frac{3m_2}{m_1} \operatorname{sech}^2 \frac{x}{2h} \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \quad ,$$

$$\hat{v} \cong (\ell-\lambda)^{3/2} 3cu_\infty [\ell^{-1} - \rho_\infty^{-1} F(\eta)] \left(- \frac{3m_2}{m_1}\right)^{3/2} \operatorname{sech}^2 \frac{x}{2h} \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \times \tanh \frac{x}{2h} \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \quad .$$

If the equation of state for a perfect gas

$$\hat{p} = R\hat{\rho}\hat{T}$$

is used, where R is the gas constant, and $\hat{T}$ the absolute temperature, then

$$\hat{T} \cong \frac{\tilde{p}_0}{R\tilde{\rho}_0} \rho^{n-1} - (\ell-\lambda) \frac{\tilde{p}_0}{R\tilde{\rho}_0} (1-n^{-1}) \rho_\infty^{-1} F(\eta) \frac{3m_2}{m_1} \operatorname{sech}^2 \sqrt{- \frac{m_2}{m_0} (\ell-\lambda)} \quad .$$

4. Discussion.

From (3.16) we see that m_0 is always negative; if $\ell^{-1}-\rho_\infty^{-1}((\eta) \equiv 0$, then $m_0 = 0$, and also $f_1 \equiv 0$ as seen from (3.11), and no solitary wave exists. It is evident that $m_2 = F(0) = \int_0^{\eta_s} \frac{\rho_\infty(\eta)}{u_\infty^2(\eta)} d\eta$ is always positive. As observed from the solution for $a(\sigma)$, $-\frac{m_2}{m_0}(\ell-\lambda)$ must be positive to ensure the existence of a solitary wave, and it follows that $\ell-\lambda > 0$ and $c^2 > \ell^{-1}gh = F(0)\frac{\tilde{p}_0}{\tilde{\rho}_0}$. In Appendix II, it will be shown that m_1 is always negative. The wave type of the solitary wave is determined by the sign of $\ell^{-1}-\rho_\infty^{-1}F(\eta)$ as seen from the expression for $\hat{f}$ in (3.17). We have a wave of elevation (depression) if $\ell^{-1}-\rho_\infty^{-1}F(\eta) > 0$ (< 0). Let us write $\ell^{-1}-\rho_\infty F(\eta)$ as $\frac{1}{\rho_\infty}[F(0)\rho_\infty-F(\eta)]$. It is easily seen that $g(\eta) = F(0)\rho_\infty-F(\eta) = 0$ when $\eta = 0,\eta_s$. In order to determine whether there are any zeros of $g(\eta)$ in $0 < \eta < \eta_s$, we established some criteria by considering the points of intersection of $\rho_\infty F(0)$ and $F(\eta)$ other than $\eta = 0,\eta_s$. We shall first find the slopes of $\rho_\infty F(0)$ and $F(\eta)$ at $\eta = 0$, $\eta=\eta_s$. Now $\rho_\infty'(\eta)F(0) = -\frac{1}{n}\rho_\infty^{-n+2}(\eta)F(0)$, $F'(\eta) = -\rho_\infty u_\infty^{-2}$.

At $\eta = 0$,

$$\left.\begin{aligned} \rho_\infty'(0)F(0) &= -\frac{1}{n}F(0) , \\ F'(0) &= -u_\infty^{-2}(0) , \end{aligned}\right\} \quad \infty > n \geq 1 ;$$

and as $\eta \to \eta_s$,

$$\begin{aligned} \rho_\infty'(\eta)F(0) &\to -\infty , && \infty > n > 2 , \\ &\to -\tfrac{1}{2}F(0) , && n = 2 , \\ &\to 0 , && 2 > n \geq 1 , \\ F'(\eta) &\to 0 , && n \geq 1 . \end{aligned}$$

Therefore, for $n > 1$, $F(\eta) < \rho_\infty F(0)$ near $\eta = \eta_s$. If $F'(0) > \rho_\infty'(0)F(0)$, then $F(\eta) > \rho_\infty F(0)$ near $\eta = 0$. It follows that there must exist at least one zero of $g(\eta)$ in $0 < \eta < \eta_s$. We may write the condition $F'(0) > \rho_\infty'(0)F(0)$ for $n > 1$ as

$$u_\infty^2 F(0) > n , \qquad \text{for } n > 1.$$

For $n = 1$, and η near $\eta_s = \infty$,

$$\rho_\infty'(\eta)F(0) = - \rho_\infty(\eta)F(0) ,$$

$$F'(\eta) \cong - \rho_\infty(\eta)u_\infty^{-2}(\eta_s) .$$

If we assume that

$$u_\infty^2(\eta_s)F(0) > 1 \quad (< 1),$$

in addition to $u_\infty^2(0)F(0) > 1$ (< 1), there also exists at least one zero of $F(0)\rho_\infty - F(\eta)$ in $0 < \eta < \eta_s$. The stream lines in $0 < \eta < \eta_s$ along which $\hat{f}_1 = 0$ are called "neutral stream lines." Since $\hat{f}_1 = 0$ at $\eta = 0$, there must exist one extremum of $\hat{f}_1$ in $0 < \eta < \eta_s$ if a neutral line occurs. Therefore, "internal waves" may also be observed in a running gas. Since

for $n > 1$,

$$\lim_{\eta \to \eta_s} \rho_\infty^{-1} F(\eta) = 0 ;$$

for $n = 1$,

$$\lim_{\eta \to \eta_s} \rho_\infty^{-1} F(\eta) = u_\infty^{-2}(\eta_s) ,$$

$$\lim_{\eta \to \eta_s} [\rho_\infty^{-1} F(\eta)]_\eta = 0 ,$$

$$\lim_{\eta \to \eta_s} [\rho_\infty^{-1} F(\eta)]_{\eta\eta} = 2\, u_\infty^{-3}(\eta_s) u_\infty'(\eta_s) ,$$

$\hat{f}_1$ is an absolute maximum at $\eta = \eta_s$ for $n > 1$, and is a relative maximum (minimum) at $\eta = \eta_s$ if $u'_\infty(\eta_s) \neq 0$ and $u_\infty(\eta_s)u'_\infty(\eta_s) > 0$ (< 0) for $n = 1$.

We shall compute a simple example, pedagogical as it may be, to illustrate what we have discussed. Let $n = 2$, and

$$\rho_\infty = (1 - \frac{1}{2}\eta)$$

by (2.2). We assume that

$$u_\infty = (1 + 3\eta)^{-1/2} .$$

Since

$$u_\infty^{-2}(0) \int_0^2 \rho_\infty u_\infty^{-2}\, d\eta = 3 > 2 ,$$

there exists a neutral stream line in $0 < \eta < 2$. Now

$$g(\eta) = -\frac{1}{2}(\eta^3 - \frac{5}{2}\eta^2 + \eta) ,$$

which has zeros at $\eta = 0$, $1/2$ and 2. The minimum of $F(0) - \rho_\infty^{-1}F(\eta)$ equal to $-1/16$ occurs at $\eta = 1/4$, and the maximum equal to 3 occurs at the free surface $\eta = 2$. The solitary wave is a wave of depression in $0 < \eta < 1/2$, and becomes one of elevation in $1/2 < \eta \leq 2$.

In the following we consider some special cases of our solution.

Case I. ρ_∞ = constant.

This case, in fact, corresponds to the one of $n \to \infty$. We have $\rho_\infty = 1$ and $\eta_s = 1$ by normalization. Then from (3.16),

$$m_0 = -\int_0^1 \ell u_\infty^2(\eta)[\ell^{-1} - F(\eta)]^2\, d\eta \ ,$$

$$m_1 = -3\ell \int_0^1 u_\infty^{-4}\, d\eta \ ,$$

$$m_2 = \ell^{-1} \ , \qquad \ell = [\int_0^1 u_\infty^{-2}(\eta)\, d\eta]^{-1} \ .$$

As shown in Appendix II,

$$m_0 = \ell \int_0^1 \int_0^{\eta} \int_0^{\eta''} u_\infty^{-2}(\eta)\, u_\infty^2(\eta')\, u_\infty^{-2}(\eta'')\, d\eta\, d\eta'\, d\eta''$$

$$- \int_0^1 \int_0^{\eta} u_\infty^2(\eta)\, u_\infty^{-2}(\eta')\, d\eta\, d\eta' \ .$$

We find from (3.17) that, at $\eta = 1$,

$$f(x,1) \cong h + h(\lambda^{-1} - \ell^{-1})(\lambda\ell^{-2})[\int_0^1 u_\infty^{-4}\, d\ ]^{-1}$$

$$\operatorname{sech}^2 \frac{x}{2h} [\lambda(\lambda^{-1} - \ell^{-1})/(-m_0)]^{1/2} \ .$$

Since $\lambda\ell^{-1} = 1 + O(\varepsilon)$, the above equation becomes

$$f(x,1) \cong h + h(\lambda^{-1} - \ell^{-1})[\lambda \int_0^1 u_\infty^{-4}\, d\eta]^{-1} \operatorname{sech} \frac{x}{2h} [(\lambda^{-1} - \ell^{-1})/(-\ell^{-1} m_0)]^{\frac{1}{2}},$$

which is the result obtained by Benjamin for the solitary wave on running water if we identify λ with g in the equations (32) and (34) of [3].

Case II. u_∞ = constant.

Without loss of generality, we take $u_\infty \equiv 1$. Making use of (2.2), we have from (3.16)

$$m_0 = \frac{-2(n-1)^2}{(2n-1)(3n-2)}\ ,$$ (1)

$$m_1 = \frac{-3n+1}{n}\ ,$$

$$m_2 = 1\ , \qquad \ell = 1\ .$$

As $n \to \infty$, $m_0 \to -1/3$, $m_1 \to -3$. We have the same solitary wave equation for a liquid of constant density obtained by Peters and Stoker if we set $r = 0$, $\ell = 1$, and $\rho = 1$ in the equation (4.20) of [2]. As $n = 1$, $m_0 = 0$, and no solitary wave exists.

(1) The coefficients m_0, m_1 and m_2 obtained below should replace those given in Part I of [4].

Appendix I.

The equations for the first approximation are

(I.1) $$G_\infty u_{1\sigma} = - f_{0\eta} p_{1\sigma} + f_{1\sigma} p_{0\eta} \quad ,$$

(I.2) $$0 = \rho_0 f_{0\eta} - \ell(\rho_0 f_{1\eta} + \rho_1 f_{0\eta}) - p_{1\eta} \quad ,$$

(I.3) $$\rho_1 u_0 f_0 + \rho_0 u_1 f_0 + \rho_0 u_0 f_1 = 0 \quad ,$$

(I.4) $$p_1 = \ell n \rho_1 \rho_0^{n-1} - \rho_0^n \quad ,$$

subject to

$$f_1(\sigma, 0) = 0 \ , \qquad\qquad p_1(\sigma, \eta_s) = 0 \ .$$

From (3.5) and (I.3), we have

(I.5) $$G_\infty u_1 = - u_\infty^2(\rho_1 f_{0\eta} + \rho_\infty f_{1\eta}) \ ,$$

where $f_{0\eta} = 1$. Multiplying the left-hand side of (I.2) by u_∞^2, and substituing the above equation for $(\rho_1 + \rho_\infty f_{1\eta})$, we obtain

(I.6) $$0 = G_\infty + \ell G_\infty u_1 - u_\infty^2 p_{1\eta} \ .$$

Then by differentiating (I.6) with respect to σ and η successively, it is found that

(I.7) $$u_\infty^2 p_{1\sigma\eta} = \ell G_\infty u_{1\sigma} \quad ,$$

(I.8) $$(u_\infty^2 p_{1\sigma\eta})_\eta = \ell (G_\infty u_{1\sigma})_\eta \quad .$$

Making use of (I.7), we obtain from (I.1) and (I.5) that

(I.9) $$f_{1\sigma} = p_{0\eta}^{-1}(u_\infty^2 p_{1\sigma\eta} + p_{1\sigma}) \quad ,$$

(I.10) $$f_{1\sigma\eta} = \rho_0^{-1}(-\ell^{-1} p_{1\sigma\eta} - \rho_{1\sigma}) \quad .$$

Now differentiating both sides of (I.1) with respect to η and making use of (I.9) and (I.10), we have

$$\ell^{-1}(u_\infty^2 p_{1\eta\sigma})_\eta = - p_{1\sigma\eta} + p_{0\eta}\rho_0^{-1}(-\ell^{-1}p_{1\sigma\eta} - \rho_{1\sigma})$$

$$+ p_{0\eta\eta}p_{0\eta}^{-1}(\ell^{-1}u_\infty^2 p_{1\sigma\eta} + p_{1\sigma}) \quad .$$

However, from (3.4) and (I.4),

$$p_{0\eta} = -\ell\rho_0 \quad ,$$

$$- p_{0\eta}\rho_0^{-1}\rho_{1\sigma} = - p_{0\eta\eta}p_{0\eta}^{-1}p_{1\sigma} \quad ,$$

it follows that

(I.11) $$(u_\infty^2 p_{1\eta\sigma})_\eta = \frac{\rho_\infty'}{\rho_\infty}(u_\infty^2 p_{1\sigma\eta}) \quad .$$

From the equations (3.12) for the second order approximation, we can easily obtain

$$G_\infty u_{2\sigma} = - f_{0\eta}p_{2\sigma} + f_{2\sigma}p_{0\eta} + g_1(\sigma,\eta) \quad ,$$

$$0 = - g_2(\sigma,\eta) + \ell G_\infty u_2 - u_\infty^2 p_{2\eta} \quad ,$$

$$G_\infty u_2 = - u_\infty^2(\rho_2 f_{0\eta} + \rho_0 f_{2\eta}) + g_3(\sigma,\eta)$$

where g_1, g_2 and g_3 are defined in (3.13). The above equations are similar to (I.1), (I.6) and (I.5), and we can obtain (3.13) by the same procedure used in deriving (I.11).

Appendix II.

We recall that the condition for $a(\sigma)$ is

$$-\int_0^{\eta_s} F(\eta')g_4(\sigma,\eta')\rho_\infty^{-1}(\eta')\,d\eta' + F(0)g_{2\sigma}(\sigma,0) - g_1(\sigma,0) = 0.$$

By a straightforward substitution of the results for the zero-th and first approximation in the above equation, we have

$$m_0 a'''(\sigma) + m_1 a'(\sigma)a(\sigma) + m_2 a'(\sigma) = 0 ,$$

where

$$m_0 = -\int_0^{\eta_s} \rho_\infty^{-1}F(\eta)[\ell^{-1}\rho_\infty u_\infty^4 - u_\infty^4 F(\eta)]_\eta\, d\eta$$

$$+\int_0^{\eta_s} F(\eta)\,(\rho_\infty^{-1}\, u_\infty^{-2} + \rho_\infty'\rho_\infty^{-2})[\ell^{-1}\rho_\infty u_\infty^4 - u_\infty^4 F(\eta)]\,d\eta$$

$$m_1 = -\int_0^{\eta_s} \rho_\infty^{-1}F(\eta)2\ell(u_\infty^{-2}\rho_\infty)_\eta\, d\eta$$

$$+\int_0^{\eta_s} F(\eta)(\rho_\infty^{-1}\ell u_\infty^{-2} + \rho_\infty'\rho_\infty^{-2})2\ell(u_\infty^{-2}\rho_\infty)\,d\eta$$

$$-\,2u_\infty^{-2}(0) - \int_0^{\eta_s} 2F(\eta)\ell^2 u_\infty^{-2}[(-u_\infty^2\rho_\infty'\rho_\infty^{-2}F^2(\eta) -$$

$$-\,F(\eta)]\,\frac{\rho_\infty^{-n}}{n} + u_\infty^{-2}]\,d\eta$$

$$-\int_0^{\eta_s} F(\eta)\ell\rho_\infty'\rho_\infty^{-2}(-\ell\rho_\infty^{-2}\rho_\infty' F^2(\eta) - \rho_\infty u_\infty^{-2})\,d\eta$$

$$+\int_0^{\eta_s} \rho_\infty^{-1}F(\eta)\ell[-\ell\rho_\infty^{-2}\rho_\infty' F^2(\eta) - \rho_\infty u_\infty^{-2}]_\eta\,d\eta\, +$$

$$+ [-\ell\rho_\infty'(0)F^2(0)-u_\infty^{-2}(0)] - \int_0^{\eta_s} \ell^2 \frac{(n-1)}{n^2} \rho_\infty^{-2n}F^3(\eta)d\eta,$$

$$m_2 = F(0)+\int_0^{\eta_s} \rho_\infty'\rho_\infty^{-1}F(\eta)\, d\eta - \int_0^{\eta_s} F(\eta)\ (\ell u_\infty^{-2}+\rho_\infty'\rho_\infty^{-1})\ d\eta$$

$$- \int_0^{\eta_s} F(\eta)\ell\rho_\infty'(\ell^{-1}\rho_\infty^{-1}-\rho_\infty^{-2}F(\eta))\ d\eta$$

$$+ \int_0^{\eta_s} F(\eta)\rho_\infty^{-1}\ell(\ell^{-1}\rho_\infty-F(\eta))_\eta\ d\eta$$

$$+ \int_0^{\eta_s} \ell n^{-1}\rho_\infty^{-n}F^2(\eta)\ d\eta\ .$$

$$\ell = F^{-1}(0)\ .$$

Here we note in passing that all the integrals are proper since $F(\eta) = O(\rho_\infty^n)$ as $\eta \to \eta_s$.

The coefficients m_0, m_1 and m_2 can be greatly simplified by showing that many terms are, in fact, equal but of opposite signs. First we consider m_0, which can be written as

$$m_0 = \int_0^{\eta_s} F(\eta)u^2(1 - \ell\rho_\infty^{-1}F^2(\eta))\ d\eta$$

$$- \int_0^{\eta_s} F(\eta)[\ell^{-1}u_\infty^4 - u_\infty^4\rho_\infty^{-1}F(\eta)]_\eta\ d\eta\ .$$

By integration by parts of the second member on the right-hand side of the above equation, we have

$$m_0 = \int_0^{\eta_s} F(\eta)u_\infty^2(1 - \ell\rho_\infty^{-1}F(\eta))\, d\eta$$

$$- \int_0^{\eta_s} \rho_\infty u_\infty^2(\ell^{-1} - \rho_\infty^{-1}F(\eta))\, d\eta$$

$$= - \int_0^{\eta_s} \ell^{-1}u_\infty^2\rho_\infty(1 - \ell\rho_\infty^{-1}F(\eta))^2\, d\eta \; .$$

For $\rho_\infty = 1$,

$$m_0 = - \ell \int_0^1 F^2(\eta)u_\infty^2\, d\eta + 2\int_0^1 F(\eta)u_\infty^2\, d\eta - \ell^{-1}\int_0^1 u_\infty^2\, d\eta \; ,$$

where $F(\eta) = \int_\eta^1 u_\infty^{-2}\, d\eta$.

By integration by parts, and noting that $\ell^{-1} = \int_0^1 u_\infty^{-2}(\eta)\, d\eta$,

we have

$$- \ell \int_0^1 F^2(\eta)u_\infty^2\, d\eta = - \int_0^1 u_\infty^{-2}(\eta)\, d\eta \int_0^1 u_\infty^2(\eta')\, d\eta' + \int_0^1\int_0^\eta u_\infty^2(\eta)u_\infty^{-2}(\eta')\, d\eta\, d\eta'$$

$$+ \ell \int_0^1\int_0^\eta\int_0^{\eta'} u_\infty^{-2}(\eta)u_\infty^2(\eta')u_\infty^{-2}(\eta'')\, d\eta\, d\eta'\, d\eta'',$$

and also we may write

$$2\int_0^1 F(\eta)u_\infty^2\, d\eta = 2\int_0^1 \left[\int_0^1 u_\infty^{-2}(\eta')\, d\eta' - \int_0^\eta u_\infty^{-2}(\eta')\, d\eta'\right]u_\infty^2(\eta)\, d\eta \; .$$

Therefore,

$$m_0 = \ell\int_0^1\int_0^\eta\int_0^{\eta'} u_\infty^{-2}(\eta')u_\infty^2(\eta')u_\infty^{-2}(\eta'')\, d\eta\, d\eta'\, d\eta''$$

$$- \int_0^1\int_0^\eta u_\infty^2(\eta)\, u_\infty^{-2}(\eta')\, d\eta\, d\eta' \; .$$

Next we consider m_1. By rearranging and combining the terms, we obtain

$$m_1 = -\int_0^{\eta_s} 3\ell F(\eta)(u_\infty^{-2})_\eta \, d\eta - 3u_\infty^{-2}(0) - \rho_\infty'(0)F(0)$$

$$+\int_0^{\eta_s} 2\ell^2 \frac{\rho_\infty^{-n-2}}{n} \rho_\infty' F^3(\eta)\, d\eta + \int_0^{\eta_s} \ell^2 \frac{(n-1)}{n^2} \rho_\infty^{-2n} F^3(\eta)\, d\eta$$

$$-\int_0^{\eta_s} \ell^2 F(\eta)[\rho_\infty^{-3}\rho_\infty' F^2(\eta)]_\eta \, d\eta + \int_0^{\eta_s} 2\ell^2 \frac{\rho_\infty^{-n}}{n} F^2(\eta) u_\infty^{-2}\, d\eta .$$

In the following, we shall show that the sum of the last four integrals on the right-hand side of the above expression is equal to zero. First we note that

$$\rho_\infty' = -\frac{1}{n}\rho_\infty^{2-n} ,$$

and

$$\int_0^{\eta_s} 2\ell^2 \frac{\rho_\infty^{-n-2}}{n} \rho_\infty' F^3(\eta)\, d\eta - \int_0^{\eta_s} \ell^2 \frac{(n-1)}{n^2} \rho_\infty^{-2n} F^3(\eta)\, d\eta$$

$$= -\int_0^{\eta_s} \ell^2 \frac{(n+1)}{n^2} \rho_\infty^{-2n} F^3(\eta)\, d\eta .$$

The term

$$-\int_0^{\eta_s} \ell^2 F(\eta)[\rho_\infty^{-3}\rho_\infty' F^2(\eta)]_\eta \, d\eta$$

$$= -\int_0^{\eta_s} \ell^2 F^3(\eta)(\rho_\infty^{-3}\rho_\infty')_\eta \, d\eta - \int_0^{\eta_s} \eta^2 \rho_\infty^{-3}\rho_\infty' F(\eta)$$

$$[F^2(\eta)]_\eta \, d\eta$$

$$= \int_0^{\eta_s} \ell^2 \frac{(n+1)}{n^2} \rho_\infty^{-2n} F^3(\eta)\, d\eta - \int_0^{\eta_s} 2\ell^2 \frac{\rho_\infty^{-n}}{n} F^2(\eta) u_\infty^{-2} d\eta .$$

Hence the sum of the last four integrals is equal to zero, and

$$m_1 = -\int_0^{\eta_s} 3\ell F(\eta)(u_\infty^{-2})_\eta \, d\eta - 3u_\infty^{-2}(0) - \rho_\infty'(0)F(0)$$

$$= -3\ell \int_0^{s} \rho_\infty u_\infty^{-4} \, d\eta \, d\eta - \rho_\infty'(0)F(0) \ ,$$

by integration by parts. In the following, we shall show that m_1 is negative. Let us write

$$m_1 = -3\ell\left[\int_0^{\eta_s} \rho_\infty u_\infty^{-4} \, d\eta + \frac{1}{3}\rho_\infty'(0)\left(\int_0^{\eta_s} \rho_\infty u_\infty^{-2} \, d\eta\right)^2\right] ,$$

it will suffice to show

$$\left(\int_0^{\eta_s} \rho_\infty u_\infty^{-2} \, d\eta\right)^2 \leq \left(\int_0^{\eta_s} \rho_\infty u_\infty^{-4} \, d\eta\right) ,$$

since $\rho_\infty'(0) = -1/n$ where $n \geq 1$.

However by Schwartz' inequality,

$$\left(\int_0^{\eta_s} \rho_\infty u_\infty^{-2} \, d\eta\right)^2 \leq \int_0^{\eta_s} \rho_\infty u_\infty^{-4} \, d\eta \int_0^{\eta_s} \rho_\infty \, d\eta = \int_0^{\eta_s} \rho_\infty u_\infty^{-4} \, d\eta \ ,$$

where $\int_0^{\eta_s} \rho_\infty \, d\eta = 1$, and we have $m_1 < 0$.

Finally we consider m_2. By replacing $(\ell^{-1}\rho_\infty - F(\eta))_\eta$ by $\ell^{-1}\rho_\infty' + \rho_\infty u_\infty^{-2}$, and noting that

$$\rho_\infty^{-2}\rho_\infty' = -\frac{\rho_\infty^{-n}}{n} \ ,$$

It is easily seen that

$$m_2 = F(0) \ .$$

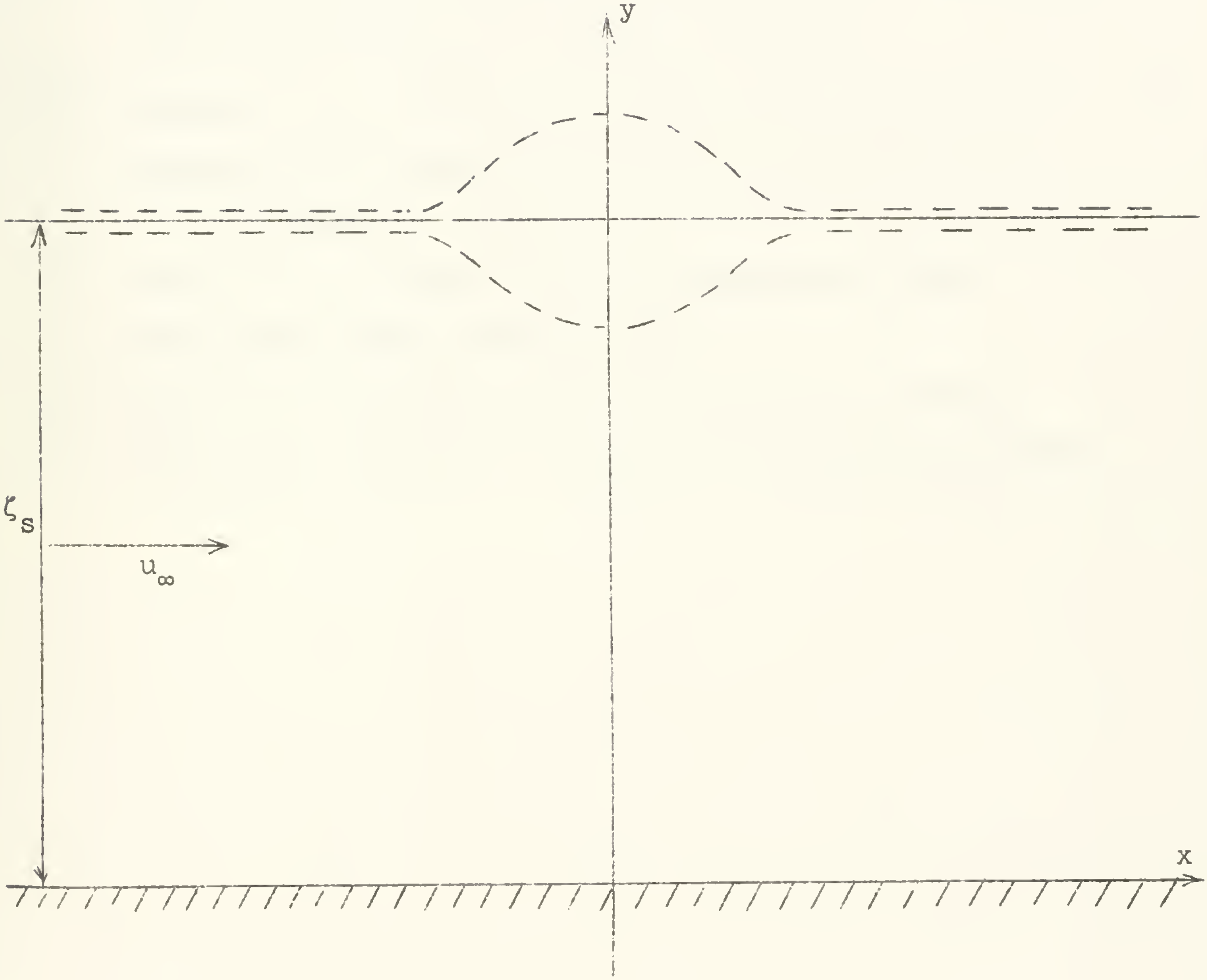

Figure 1

References

[1] Stoker, J. J., Water Waves, 1957, Interscience, pp. 342-351.

[2] Peters, A. S., and Stoker, J. J., Solitary Waves in Liquids having Non-constant Density, Comm. Pure and Appl. Math., Vol. XIII, No. 1, 1960, pp. 115-164.

[3] Benjamin, J. B., The Solitary Wave on a Stream with an Arbitrary Distribution of Vorticity, J. Fluid, Mech., Vol. 12, 1962, pp. 97-116.

[4] Shen, M. C., Solitary Waves in Compressible Media, NYU Research Report IMM-NYU 325.

[5] Meyer, R. E., Theory of Characteristics of Inviscid Gas Dynamics, Encycl. of Physics, 9, Heidelberg-Springer.

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Solitary waves in running polytropic gases of finite or infinite height are studied. Explicit expressions for the critical speed and the solution of the solitary waves are obtained by a perturbation scheme applies to the nonlinear equations. It is found that internal waves may also be observed in a running gas, and change of wave type may occur in the medium. The results obtained are expressed in terms of simple integrals of density and velocity distributions of the medium in the state of equilibrium, and conform with solutions for liquids of constant density as special cases.

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